ON CENTRAL EXTENSIONS OF PREPROJECTIVE ALGEBRAS

PAVEL ETINGOF, FRÉDÉRIC LATOUR, AND ERIC RAINS

Dedicated to the 70th birthday of Ernest Vinberg

1. Introduction

Let Q be a quiver of ADE type. Let $\overline{Q}$ be the double of Q, and P the path algebra of $\overline{Q}$ over $\mathbb{C}$. The paper [ER] attaches to Q a centrally extended preprojective algebra $A = A^\mu$, which is the quotient of $P[z]$ by the relation $\sum_{a\in Q}[a, a^*] = z(\sum \mu_i e_i)$, where $\mu = (\mu_i)$ is a regular weight (for the root system attached to Q), and e_i are the vertex idempotents in P.[1] It is shown in [ER] that the algebra A has nicer properties than the ordinary Gelfand-Ponomarev preprojective algebra $A_0 = A/(z)$ of Q; in particular, the deformed version $A(\lambda)$ of $A = A(0)$ is flat, while this is not the case for A_0. The paper [ER] also shows that A is a Frobenius algebra, and computes the Hilbert series of A. Finally, [ER] links the algebra A with cyclotomic Hecke algebras of complex reflection groups of rank 2.

The goal of this paper is to continue to study the rich structure of the algebra A. In particular, we show that for generic μ (and specifically for $\mu = \rho$) the algebra A has a unique trace functional, and compute the structure of the center Z of A and the trace space $A/[A, A]$. Namely, it turns out that Z and $A/[A, A]$ are dual to each other under the trace form, and the dimension of the homogeneous subspace $(A/[A, A])[2p]$ equals the number of positive roots for Q of height $p + 1$.

We also show that the elements $z^s(\sum \phi_i e_i)$ span $A/[A, A]$, and determine when such an element maps to zero in $A/[A, A]$ (i.e. sits in $[A, A]$). The answer is given in terms of the structure of the maximal nilpotent subalgebra $\mathfrak{n}$ of the simple Lie algebra $\mathfrak{g}$ attached to Q, which demystifies the equality between $\dim(A/[A, A])[2p]$ and the number of positive roots for Q of height $p + 1$.

Acknowledgments. P.E. thanks George Lusztig for a useful discussion. The work of P.E. was partially supported by the NSF grant DMS-0504847 and the CRDF grant RM1-2545-MO-03. F.L. thanks the IHÉS in Bures-sur-Yvette, France. E.R. was supported in part by NSF grant DMS-0401387.

2. Preliminaries and some results

2.1. **Preliminaries.** We recall some definitions and notation from [ER].

[1]This algebra is denoted in [ER] by Π_0^μ.

Let Q be a quiver of ADE type. Let I be the set of vertices of Q, and $r = |I|$.

Consider the root system $\mathcal{R}$ attached to Q. Let $\alpha_j, j \in I$, be the simple roots. Let $\omega_j, j \in I$, be the fundamental weights. Let $\rho = \sum \omega_i$. If α is a positive root, then the height of α is the number of simple roots occuring in the decomposition of α; it equals to the inner product (ρ, α). Let h be the Coxeter number of $\mathcal{R}$. Let N be the number of positive roots in $\mathcal{R}$. Recall that $N = hr/2$.

Let $\mathfrak{g}$ be the simple Lie algebra whose Dynkin diagram is Q. Fix a polarization $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$, where $\mathfrak{n}_\pm$ are the nilpotent subalgebras, and $\mathfrak{h}$ the Cartan subalgebra. For brevity we will denote $\mathfrak{n}_-$ by $\mathfrak{n}$. The Lie algebra $\mathfrak{n}$ is generated by elements $F_i, i \in I$, subject to the Serre relations.

Let R be the algebra of complex-valued functions on I, and $e_i, i \in I$, be the primitive idempotents of this algebra. Let $\overline{Q}$ be the double of Q. Let V be the R-bimodule spanned by the edges of $\overline{Q}$. Let $P = T_R V$ be the path algebra of the doubled quiver $\overline{Q}$ (the tensor algebra over R of the bimodule V). Let $\mu = \sum_{i \in I} \mu_i \omega_i \in \mathfrak{h}^*$ be a regular weight (i.e. the inner product $(\mu, \alpha) \neq 0$ for any root $\alpha \in \mathcal{R}$). Define the centrally extended preprojective algebra $A = A^\mu$ of Q, which is the quotient of $P[z]$ (where z is a central variable) by the relation

$$\sum_{a \in Q} [a, a^*] = z(\sum_{i \in I} \mu_i e_i).$$

Note that if $\mu = \rho$ then this relation takes an especially simple form

$$\sum_{a \in Q} [a, a^*] = z.$$

Also, let $A_0 := A/(z)$ be the usual preprojective algebra of Q (it is the quotient of P by the relation $\sum_{a \in Q} [a, a^*] = 0$.)

Define the deformed centrally extended preprojective algebra $A(\lambda) = A^\mu(\lambda)$ to be the quotient of the path algebra $P[z]$ by the defining relation

$$\sum_{a \in Q} [a, a^*] = \sum_{i \in I} (\mu_i z + \lambda_i) e_i,$$

where $\lambda = \sum_{i \in I} \lambda_i \omega_i \in \mathfrak{h}^*$ is a weight. This algebra carries a natural filtration, given by $\deg(R) = 0$, $\deg(a) = \deg(a^*) = 1$, $\deg(z) = 2$. It is shown in [ER] that $A(\lambda)$ is a flat deformation of $A(0) = A$, i.e., $\mathrm{gr}(A(\lambda)) = A(0)$.

It is clear that the algebras A_0 and $A(\lambda)$ are independent on the orientation of Q, up to an isomorphism.

2.2. **The trace function on A.** From now on we assume that μ is a fixed generic weight, or $\mu = \rho$.

Recall from [ER] that A is a finite dimensional $\mathbb{Z}_+$-graded Frobenius algebra, with socle in degree $2(h-2)$, with basis $z^{h-2}e_i$.[2]

Proposition 2.1. *(i) There exists a unique up to scaling trace* $Tr : A \to \mathbb{C}$ *of degree* $2(h-2)$*, i.e. a nonzero linear functional such that* $Tr(xy) = Tr(yx)$.

(ii) The form $(x, y) := Tr(xy)$ *is nondegenerate.*

Proof. Clearly, we may assume that Q has at least two vertices. Recall that the degree 1 component $A[1]$ of A is spanned by edges a of the doubled quiver $\overline{Q}$. Also, $A[2(h-2)-1]$ is spanned by elements of the form $z^{h-3}b$, where b is an edge of $\overline{Q}$. Indeed, it follows from [ER], Section 4, that this is true for $\mu = \rho$, hence it is true for generic μ by deformation argument.

Since A is a Frobenius algebra, for every edge a we have $z^{h-3}aa^* = c_a z^{h-2}e_{head(a)}$, where c_a is a nonzero number.

But $[A, A][2(h-2)] = [A[1], A[2(h-2)-1]]$, so it is the span of $z^{h-3}[a, a^*]$ for the edges $a \in Q$, i.e. of elements $z^{h-2}(c_a e_{head(x)} - c_{a^*} e_{tail(x)})$. It is clear that these elements span a subspace of codimension 1 in $A[2(h-2)]$; thus the functional Tr is unique up to scaling. Moreover, $\mathrm{Tr}(z^{h-2}e_i)$ is clearly nonzero for any i. The proposition is proved. □

Now let Z be the center of A.

Corollary 2.2. *The inner product* (x, y) *defines a nondegenerate pairing* $Z \times A/[A, A] \to \mathbb{C}$.

Proof. The statement is well known but we give a proof for completeness. If $x \in Z$ and $y = [y_1, y_2] \in [A, A]$ then $(x, y) = \mathrm{Tr}(x[y_1, y_2]) = \mathrm{Tr}([xy_1, y_2]) = 0$. Thus the pairing in question is well defined. To show that it is nondegenerate, by Proposition 2.1 (ii), it suffices to show that $Z^\perp \subset [A, A]$, or equivalently, $Z \supset [A, A]^\perp$.

The latter statement is obvious. Indeed, if $\mathrm{Tr}(x[y_1, y_2]) = 0$ for any y_1, y_2, then $\mathrm{Tr}([x, y_1]y_2) = 0$ for any y_1, y_2, and therefore $[x, y_1] =$ for all y_1, implying $x \in Z$. □

Let $p(t) = \sum \dim(A/[A, A])[m]t^m$ be the Hilbert polynomial of $A/[A, A]$, and $p_*(t) = \sum \dim Z[m]t^m$ be the Hilbert polynomial of Z.

Corollary 2.3. *The polynomials* p, p_* *are palindromes of each other, i.e.* $p_*(t) = t^{2(h-2)}p(1/t)$.

2.3. **The spaces Z and $A/[A, A]$ as $\mathbb{C}[z]$-modules.** Let E be the subspace of A spanned by elements $z^j e_i$. Obviously, it has dimension $(h-1)r$. The Hilbert polynomial of E is $\frac{1-t^{2h-2}}{1-t^2}r$.

Proposition 2.4. *The natural map* $\psi : E \to A/[A, A]$ *is surjective.*

[2]Note that the elements $z^{h-2}e_i$ may vanish for special regular μ.

Proof. It is shown in [MOV], Section 4, that $A_0/[A_0, A_0]$ is freely spanned by the idempotents e_i. This implies that if $x \in A$ is an element of positive degree d then there exists a homogeneous element $y \in A$ of degree $d-2$ such that $x - zy \in [A, A]$. Thus the statement follows by induction in d. $\square$

Note now that $A/[A, A]$ and Z are naturally $\mathbb{C}[z]$-modules, and the pairing $(,)$ between them is invariant in the sense that the operator of multiplication by z is selfadjoint.

Corollary 2.5. *The $\mathbb{C}[z]$-module $A/[A, A]$ is minimally generated by e_i.*

Proof. Indeed, $A/[A, A]$ is a quotient of E by the submodule $E \cap [A, A]$, which shows that it is generated by e_i. The minimality of this set of generators is obvious. $\square$

Thus we see that the operator z in $A/[A, A]$ and Z is a direct sum of r nilpotent Jordan blocks, of some sizes $m_1 \le m_2 \le ... \le m_r$, and $p(t) = \sum_{i=1}^{r} \frac{1-t^{2m_i}}{1-t^2}$.

3. The main theorem

Let N_p be the number of positive roots for Q of height $p + 1$.

One of our main results is the following theorem.

Theorem 3.1. *(i)* $\dim Z = \dim(A/[A, A]) = N$.

(ii) The sizes m_i of the Jordan blocks of z on Z and $A/[A, A]$ are the exponents of the root system attached to Q. In other words, we have $\dim(A/[A, A])[2p] = N_p$ *for all $p \ge 0$.*

The proof of Theorem 3.1 is given in the next two subsections.

3.1. **The lower bound.**

Proposition 3.2. $\dim Z \ge N$.

Proof. According to [ER], for generic λ the algebra $A(\lambda)$ is semisimple with irreducible representations V_α corresponding to positive roots α. This implies that the center $Z(\lambda)$ of $A(\lambda)$ is a semisimple algebra of dimension N. Since $\mathrm{gr}(A(\lambda)) = A$, we have $\mathrm{gr}(Z(\lambda)) \subset Z$, and we get the desired inequality. $\square$

3.2. **The upper bound.** We have $\sum_p N_p = N$. Theorefore, by Proposition 3.2, to prove Theorem 3.1, it suffices to show that $\dim(A/[A, A])[2p] \le N_p$ for all $p \ge 0$.

We do it case by case, following the idea of the argument of [MOV], Section 4. Since we need to establish the result for generic μ, it suffices to consider the case $\mu = \rho$.

Case 1: type A_n. In this case $N_p = \max(n - p, 0)$. Denote the corresponding algebra A by A^n, and let us prove the desired statement by induction in n.

The base of induction ($n = 1$) is obvious, so let us perform the induction step. Assume that the statement is known for A^{n-1}, and let us prove it for A^n ($n \geq 2$).

Let $J = Ae_nA$ be the ideal in A^n spanned by paths passing through the end-vertex n of the Dynkin diagram. Then $A^n/J = A^{n-1}$. Thus, using the induction assumption, we see that

$$\dim A^n/(J + [A^n, A^n])[2p] \leq \max(n - 1 - p, 0).$$

Thus to establish the induction step (i.e. to show that $\dim(A^n/[A^n, A^n]) \leq n - p$), it suffices to show that $\dim(J/(J \cap [A^n, A^n]))[2p] \leq 1$ for $p \leq n - 1$ and is zero if $p \geq n$.

Define the algebra $B_n := e_nA^ne_n$. It is easy to see that the natural map $\phi : B_n \to J/(J \cap [A^n, A^n])$ is surjective. On the other hand, it is easy to check that B_n is a commutative n-dimensional algebra: $B_n = \mathbb{C}[z]/(z^n)$. Thus the desired statement follows.

Case 2: types D, E. Let $*$ be the nodal vertex of the Dynkin diagram, and $J = Ae_*A$. Then $A/J = A^{\ell_1} \oplus A^{\ell_2} \oplus A^{\ell_3}$, where ℓ_1, ℓ_2, ℓ_3 are the lengths of the three legs of the Dynkin diagram. Thus by Case 1,

$$\dim A/(J + [A, A])[2p] \leq \sum_{j=1}^{3} \max(\ell_j - p, 0).$$

So it suffices to show that

$$\dim J/(J \cap [A, A])[2p] \leq N'_p,$$

where $N'_p := N_p - \sum_{j=1}^{3} \max(\ell_j - p, 0)$ is the number of positive roots of height $p + 1$ which contain the simple root α_* in their expansion.

Define the algebra $B := e_*Ae_*$. It is easy to see that the natural map $\phi : B \to J/(J \cap [A, A])$ is surjective. Thus, it suffices to show that

$$\dim(B/[B, B])[2p] \leq N'_p.$$

According to [ER], the algebra B is generated by degree 2 elements U_1, U_2, U_3 with defining relations

$$U_1 + U_2 + U_3 = z, \ [z, U_i] = 0, \prod_{m=0}^{\ell_i} (U_i + mz) = 0, \ i = 1, 2, 3. \tag{3.1}$$

Case 2a. Type D_{n+2} ($n \geq 2$). We have $\ell_1 = \ell_2 = 1$, $\ell_3 = n - 1$. So, setting $a = U_1 + z/2$, $b = U_2 + z/2$, we have the following defining relations for B:

$$a^2 = b^2 = z^2/4, [a, z] = [b, z] = 0,$$

$$(a + b - 2z)(a + b - 3z)...(a + b - (n + 1)z) = 0.$$

Let $a_s = aba..., b_s = bab...$ (words of length s).

Lemma 3.3. *If $p < n$ then a basis of $B[2p]$ is formed by the words $a_s z^{p-s}$, $b_s z^{p-s}$, $p \geq s > 0$, and z^p. If $p = n$, then a basis of $B[2p]$ is formed by the words $a_s z^{p-s}$, $b_s z^{p-s}$, $n > s > 0$, a_n and z^n. If $p > n$, then a basis of $B[2p]$ is formed by the words $a_s z^{p-s}$, $b_s z^{p-s}$, $2n - p \geq s > 0$, and z^p.*

Proof. It is easy to see from the relations that these words are a spanning set for $B[2p]$. The fact that they are linearly independent follows from the Hilbert series formula for B given in [ER]. $\square$

Lemma 3.4. *One has* $\dim(B/[B,B])[2p] \leq N'_p$.

Proof. It is straightforward to show (by explicit inspection of the root system of type D_{n+2}) that $N'_0 = 1$, $N'_p = 3 + [p/2]$ if $1 \leq p \leq n-1$, $N'_n = 2 + [n/2]$, and $N_p = 1 + [n - p/2]$ for $p > n$, where $[x]$ is the integer part of x. For odd $s > 1$ and $p \geq s$, we have

$$a_s z^{p-s} = \frac{1}{4} b_{s-2} z^{p-s+2} + [a, b_{s-1} z^{p-s}],$$

$$b_s z^{p-s} = \frac{1}{4} a_{s-2} z^{p-s+2} + [b, a_{s-1} z^{p-s}].$$

Also, for even $s > 0$,

$$(a_s - b_s) z^{p-s} = [a, b_{s-1} z^{p-s}].$$

This together with Lemma 3.3 implies that $(B/[B,B])[2p]$ is spanned by z^p, az^{p-1}, bz^{p-1}, and $a_s z^{p-s}$ for even $s > 0$. Hence, for $p \geq 1$ we have

$$\dim(B/[B,B])[2p] \leq 3 + [p/2],$$

i.e. the Lemma is proved for $p < n$. Moreover, for $p = n$ the last relation of B implies that z^n is a linear combination of $a_s z^{n-s}$ and $b_s z^{n-s}$ for $s > 0$, which implies that

$$\dim(B/[B,B])[2n] \leq 2 + [n/2],$$

i.e. the Lemma is also proved for $p = n$.

Now let us prove the lemma for $p > n$. Let Z_B be the center of B. The pairing (x, y) of Proposition 2.1 has degree $4n$. Therefore, by Proposition 2.1 (similarly to Corollary 2.2), it suffices to show that $\dim Z_B[2k] \leq 1 + [k/2]$ for $k < n$. In showing this, we can obviously ignore the last relation of B (which has degree $2n$). In other words, we should consider the algebra B' with generators a, b, z and relations $a^2 = b^2 = z^2/4$, $[z, a] = [z, b] = 0$. It is easy to see that a basis in B' is formed by elements $z^p(a+b)^{2q}$, $az^p(a+b)^{2q}$, $bz^p(a+b)^{2q}$, $abz^p(a+b)^{2q}$, $p, q \geq 0$, and thus the center of B' is spanned by $z^p(a+b)^{2q}$, which implies the desired inequality. $\square$

Case 2b. Types E_6, E_7, E_8. Using the presentation (3.1) of B and the Magma code by the third author [Mag], one determines, by a direct computer calculation, that $\dim(B/[B,B])[2p] = N'_p$.

Theorem 3.1 is proved.

3.3. **Derivations of A.** Theorem 3.1 implies the following result.

Corollary 3.5. *Every derivation of A which annihilates R and z is inner.*

Remark. In this corollary, we can omit the hypothesis that the derivation annihilates R. Indeed, for any derivation D of A, if we let $u_D := \sum_{i\in I} e_i D(e_i)$, then $D + \mathrm{Ad}(u_D)$ annihilates R and has the same action as D on the central element z.

Proof. We consider the complex of graded vector spaces

$$0 \to D_0 \to D_1 \to D_2 \to 0,$$

with differentials $d_i : D_i \to D_{i+1}$, where $D_0 = A^R[2]$, $D_1 = (A \otimes_R V)^R$, $D_2 = A^R$ (where the superscript R denotes the R-invariants in a bimodule, and $[i]$ denotes the shift of grading), and

$$d_0(x) = \sum_{a\in Q}([x, a] \otimes a^* - [x, a^*] \otimes a),$$

$$d_1(y \otimes b) = [y, b].$$

It is clear that these differentials have degree 0. The fact that $d_1 \circ d_0 = 0$ follows from the fact that z is a central element.

Let H_0, H_1, H_2 be the homology groups of the complex $D_\bullet$. Then we have $H_0 = Z[2]$, $H_2 = A/[A, A]$.

Let $q(t)$ be the Hilbert polynomial of H_1. Then, computing the Euler characteristic in each homogeneous component of $D_\bullet$, we obtain the following identity for Hilbert polynomials:

$$t^2 p_*(t) + p(t) - q(t) = \mathrm{Tr}((1 - Ct + t^2)h(t)),$$

where $h(t)$ is the (matrix valued) Hilbert polynomial of A, and C is the adjacency matrix of $\overline{Q}$. But it is proved in [ER] that

$$h(t) = \frac{1 - t^{2h}}{1 - t^2}(1 - Ct + t^2)^{-1}.$$

This implies that

$$q(t) = t^2 p_*(t) + p(t) - \frac{1 - t^{2h}}{1 - t^2} r.$$

Now recall that the exponents of a root system satisfy the equality $m_{r+1-i} = h - m_i$. This implies that $t^2 p_*(t) + p(t) = \frac{1-t^{2h}}{1-t^2} r$, and hence $q(t) = 0$. Thus $H_1 = 0$.

Now let D be a derivation of A which annihilates R and z. Let $x_D := \sum_{a\in Q}(Da \otimes a^* - Da^* \otimes a)$. Then $d_1 x_D = 0$. Since $H_1 = 0$, this implies that $x_D = d_0 y$, i.e. $D = \mathrm{ad} y$, as desired. The corollary is proved. $\square$

4. Relation to simple Lie algebras

The computer assisted case-by-case proof of Theorem 3.1 makes it look mysterious (especially part (ii)). The results of this section demystify this theorem, by making explicit the relation of the structure of $A/[A,A]$ with that of the maximal nilpotent subalgebra of the simple Lie algebra corresponding to Q.

4.1. **The results.** Let us color the vertices of Q white and black so that every edge connects a white vertex with a black vertex. Let ε_i be $+1$ for white vertices i and -1 for black vertices. Let $F = \sum_i \varepsilon_i F_i$ be a principal nilpotent element.

Let $h_\lambda \in \mathfrak{h}$ be the element corresponding to the weight $\lambda \in \mathfrak{h}^*$ under the standard inner product on $\mathfrak{h}^*$ normalized so that $(\alpha,\alpha) = 2$ for roots α.

The following theorem characterizes explicitly the space $E \cap [A,A]$.

Theorem 4.1. *Let $\phi_i, i \in I$ be complex numbers, and $s \geq 0$ be an integer. Then the element $z^s(\sum_i \varepsilon_i \phi_i e_i)$ is in $[A,A]$ if and only if*

$$(\mathrm{ad}(F)\mathrm{ad}(h_\mu)^{-1})^s(\sum \phi_i F_i) = 0$$

in $\mathfrak{n}$.

Note that Theorem 4.1 implies Theorem 3.1. In the proof of Theorem 4.1, given in the next subsection, we will use only part (i) of Theorem 3.1, so we obtain a new proof of Theorem 3.1, part (ii).

The result of Theorem 4.1 can be stated more explicitly as follows.

Let V_i be the space of complex-valued functions on the set of positive roots for the quiver Q of height i (i.e. sums of i simple roots). Define the operator $T_i : V_i \to V_{i+1}$ by the formula

$$(T_i f)(\alpha) = \sum_{j:(\alpha_j,\alpha)=1} \frac{f(\alpha - \alpha_j)}{(\mu, \alpha - \alpha_i)}.$$

(Note that $\alpha - \alpha_j$ is a root iff $(\alpha, \alpha_j) = 1$.)

Theorem 4.2. *Let $\phi \in V_1$, $\phi_i = \phi(\alpha_i)$. Let $s \geq 0$. Then element $z^s(\sum_i \varepsilon_i \phi_i e_i)$ is in $[A,A]$ iff $T_s T_{s-1} ... T_1 \phi = 0$.*

Proof. According to [Lu], there is a Chevalley basis $\{F_\alpha\}$ of $\mathfrak{n}$ normalized in such a way that $[F_i, F_\alpha] = \varepsilon_i F_{\alpha+\alpha_i}$ provided $\alpha + \alpha_i$ is a root. Therefore,

$$\mathrm{ad}(F)\mathrm{ad}(h_\mu)^{-1} \sum_{\beta \in \mathcal{R}:(\rho,\beta)=d} \phi_\beta F_\beta =$$

$$\sum_{\gamma \in \mathcal{R}:(\rho,\gamma)=d+1} \sum_i \frac{\phi_{\gamma-\alpha_i}}{(\mu, \gamma - \alpha_i)} F_\gamma.$$

Thus Theorem 4.1 implies Theorem 4.2. □

Corollary 4.3. *The explicit form of the trace functional for A is*

$$Tr(z^{h-2}e_i) = \varepsilon_i T_{h-2}...T_1 u_i,$$

where $u_i \in V_1$ is such that $u_i(\alpha_j) = \delta_{ij}$.

This formula can be written more explicitly as follows. Let θ be the maximal root of $\mathcal{R}$. Define a path in $\mathcal{R}$ to be a sequence of positive roots $\beta_1, \beta_2, ..., \beta_m$ such that $\beta_{i+1} - \beta_i = \alpha_{j_i}$ for some j_i. Define weight of such a path π to be

$$w_\mu(\pi) = \prod_{i=1}^{m-1} (\mu, \beta_i)^{-1}.$$

Then we get

$$\mathrm{Tr}(z^{h-2}e_i) = \varepsilon_i \sum_\pi w_\mu(\pi),$$

where the summation is taken over all paths π which start at α_i and end at θ (so they have length $h-1$). In particular, if $\mu = \rho$ then after renormalization we get

$$\mathrm{Tr}(z^{h-2}e_i) = \varepsilon_i n_i,$$

where n_i is the number of paths leading from α_i to θ.

4.2. **Proof of Theorem 4.1.** Let $W(\lambda)$ be the space of collections of polynomials $f_i, i \in I$ of degree $\leq h-2$, such that $\sum_{i \in I} f_i(z)\varepsilon_i e_i \in [A(\lambda), A(\lambda)]$.

Proposition 4.4. *Let λ be generic. Let f_i, $i \in I$, be polynomials of degree $\leq h-2$. Then $\{f_i, i \in I\}$ belongs to $W(\lambda)$ iff*

$$\sum f_i\left(-\frac{(\lambda,\alpha)}{(\mu,\alpha)}\right)\varepsilon_i(\alpha,\omega_i) = 0$$

for all positive roots α.

Proof. Let us calculate the trace of $\sum f_i(z)\varepsilon_i e_i$ in the irreducible representation V_α of $A(\lambda)$ whose dimension vector is α. Since z acts on this representation by the scalar $-\frac{(\lambda,\alpha)}{(\mu,\alpha)}$, we get the statement. □

Proposition 4.5. *Let λ be generic. Let f_i, $i \in I$, be polynomials of degree $\leq h-2$. Then $\{f_i, i \in I\}$ belongs to $W(\lambda)$ iff*

$$\sum_{i \in I} f_i(\mathrm{ad}(-h_\lambda + F)\mathrm{ad}(h_\mu)^{-1})F_i = 0.$$

Proof. The linear operator $L := \mathrm{ad}(-h_\lambda + F)\mathrm{ad}(h_\mu)^{-1}$ on $\mathfrak{n}$ has eigenvalues $-(\lambda,\alpha)/(\mu,\alpha)$, where α ranges over positive roots; let the corresponding eigenvectors be v_α. Then, if we write

$$F_i = \sum_\alpha c_i(\alpha)v_\alpha,$$

we find

$$\sum_{i\in I} f_i(\mathrm{ad}(-h_\lambda+F)\mathrm{ad}(h_\mu)^{-1})F_i = \sum_{i\in I}\sum_\alpha f_i\{(-\frac{(\lambda,\alpha)}{(\mu,\alpha)}\})c_i(\alpha)v_\alpha.$$

In particular, to prove the proposition, it will suffice to show that

$$c_i(\alpha) \propto \varepsilon_i(\alpha,\omega_i).$$

Now, by duality, $c_i(\alpha)$ can be computed as the coefficient of $E_i \in \mathfrak{n}_+$ in the expansion of the eigenvector v^*_α of the dual operator

$$L^* = \mathrm{ad}(h_\mu)^{-1}\mathrm{ad}(-h_\lambda+F)$$

on $\mathfrak{g}$ with eigenvalue $-(\lambda,\alpha)/(\mu,\alpha)$.

Let $X_\alpha \in \mathfrak{n}_+$ be the projection of v^*_α to $\mathfrak{n}_+$ along $\mathfrak{h}\oplus\mathfrak{n}_-$. Then the element

$$y := [-h_\lambda + F + \frac{(\lambda,\alpha)}{(\mu,\alpha)}h_\mu, X_\alpha]$$

must belong to the Cartan subalgebra $\mathfrak{h}$.

Recall that $\lambda \in \mathfrak{h}^*$ is generic. Therefore, if $\nu \in \mathfrak{h}^*$ is any element of the orthogonal complement of α, then there exists $\mathcal{N} \in \mathfrak{n}$ such that

$$[-h_\lambda + F + \frac{(\lambda,\alpha)}{(\mu,\alpha)}h_\mu, h_\nu + \mathcal{N}] = 0,$$

and thus $(y, h_\nu + \mathcal{N}) = 0$. It follows that $y \propto h_\alpha$; since X_α was only determined up to scale, we may as well insist that $y = h_\alpha$.

Since $X_\alpha = \sum c_i(\alpha)E_i +$ lower terms, we find that

$$h_\alpha = [F, X_\alpha]_{\mathfrak{h}} = \sum \varepsilon_i c_i(\alpha) h_{\alpha_i}$$

(where the subscript $\mathfrak{h}$ denotes the $\mathfrak{h}$-part), and thus $\varepsilon_i c_i(\alpha) = (\alpha,\omega_i)$ as required. □

Now we can finish the proof of Theorem 4.1. For this, note that by Theorem 3.1(i), the space $W(0)$ is the limit of spaces $W(\lambda)$ as $\lambda \to 0$. Therefore, Proposition 4.5 implies Theorem 4.1.

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E-mail address: `etingof@math.mit.edu`

E-mail address: `flatour@mailbox.syr.edu`

E-mail address: `rains@math.ucdavis.edu`